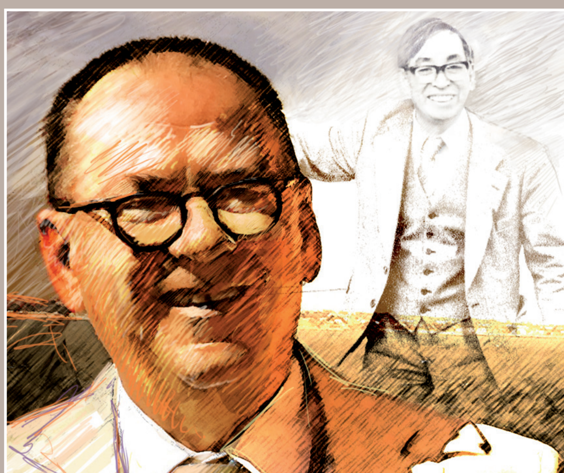
Glenn Shafer

The Reconciliation of Betting and Measure



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The Reconciliation of Betting and Measure

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This note is based on the author's talk at the Cournot Centre's conference celebrating the centennial of the birth of Kiyosi Itô, "Kiyosi Itô's Legacy from a Franco-Japanese Perspective," held November 26–27, 2015, in Tokyo. Some of the work it reports has developed since that conference was held. In the course of preparing the text, the author has benefited from advice and assistance from Bernard Bru, Therrese Goodlett, Laurent Mazliak, Jean-Philippe Touffut, Vladimir Vovk, and Adarsh Vijayaraghavan.

Summary

Three-and-a-half centuries ago, Blaise Pascal and Pierre Fermat proposed competing solutions to a classical problem in probability theory, the problem of points. Pascal looked at the paths play in the game might take. Fermat counted the combinations. The interplay between betting and measure has been intrinsic to probability ever since.

In the mid-twentieth century, this duality could be seen beneath the contrasting styles of Paul Lévy and Joseph Doob. Lévy's vision was intrinsically and sometimes explicitly game-theoretic. Intuitively, his expectations were those of a gambler; his paths were formed by outcomes of successive bets. Doob confronted Lévy's intuition with the cold rigor of measure. Kiyosi Itô was able to reconcile their visions, clothing Lévy's pathwise thinking in measure-theoretic rigor.

The reconciliation is now understood in terms of measure. But the game-theoretic intuition has been resurgent in applications to finance, and recent work shows that the game-theoretic picture can be made as rigorous as the measure-theoretic picture. In this game-theoretic picture, martingales regain their identity as capital processes and are used to define probability one and develop a purely game-theoretic version of Itô's calculus. Details are provided in my forthcoming book with Vladimir Vovk, *Game-Theoretic Foundations for Probability and Finance* [28].

Contents

1 Introduction

In 1654, Blaise Pascal and Pierre Fermat proposed competing solutions to the problem of dividing the stakes in an interrupted game, a problem that was already classical at that time.[1] Two players have put up equal amounts of money. The first to win a given number of rounds will win all the money. How should the money be divided when play is interrupted by external circumstances? The problem is not merely one of calculation; we must also agree on principles for fair division.

Pascal's solution was game-theoretic; he considered the players' capital on the different paths play might take. Fermat's was measure-theoretic; he counted cases. The duality and interplay between betting and measure has been intrinsic to probability ever since.

In the mid-20th century, the duality between betting and measure was symbolized by the contrasting visions of Paul Lévy and Joseph Doob. Lévy's vision was thoroughly and sometimes explicitly game-theoretic. His expectations were those of a gambler; his paths were formed by successive outcomes in a game. Doob confronted Lévy's intuition with the cold precision of measure. In the early 1940s, Kiyosi Itô was able to reconcile their visions, clothing Lévy's pathwise thinking in measure-theoretic rigor.

Today Lévy's picture is even more thoroughly understood in terms of measure, but the game-theoretic intuition is resurgent in applications to finance. Recent work shows that the

[1] See [25] for a review of the history of this problem in the centuries before Pascal and Fermat. The game was usually a ball game or some other competition involving skill, not a game of pure chance. The problem of dividing the stakes is usually called the problem of points in English, but this can be considered a mistranslation of the French "problème des partis". See [16], p. 93.

game-theoretic picture can be made as rigorous as the measure-theoretic picture. Since it brings Lévy's and Itô's intuitions more vividly to life, while remaining fundamentally consistent with the measure-theoretic picture, the rigorous game-theoretic picture furthers Itô's reconciliation of Doob and Lévy.

In this article, I review Pascal and Fermat's arguments, the different definitions of probability to which they lead, Lévy's game-theoretic intuition, and Itô's measure-theoretic elaboration of it. Then I discuss the relevance of recent work on game-theoretic probability that is reported in depth in two books I have co-authored with Vladimir Vovk: *Probability and Finance: It's Only a Game* (2001, [26]) and the forthcoming *Game-Theoretic Foundations for Probability and Finance* [28].

2 Pascal (betting) vs. Fermat (measure)

Pascal and Fermat agreed that the division of the stakes in an interrupted game should depend only on the possible ways the game might have continued were it not interrupted. Suppose the two players are named Peter and Paul. When the game is interrupted, Peter needs to win only one more round to win the game, but Paul needs to win two. In the story Pascal told to Fermat, each player has put up 32 pistoles. A pistole being a gold coin with purchasing power comparable to that of a few hundred dollars today, the total stakes of 64 pistoles was a substantial amount of money.

The game tree in Figure 1 shows what might have happened had Peter and Paul been able to finish their game. The numbers represent how much Paul might win: either 64 pistoles or zero pistoles. If Peter wins the first round, Paul gets zero. If Paul wins the first round and Peter wins the second, Paul gets zero.

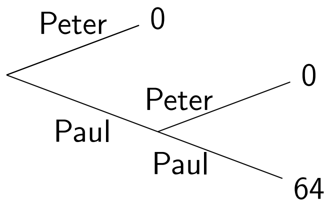

Figure 1: Game tree representing Pascal's question to Fermat in 1654

Paul gets the 64 pistoles only if he wins both rounds.

Pascal asked Fermat how many of the 64 pistoles should go to Paul. Fermat answered by noting that there would be four possible outcomes if the players played two more rounds:

1. Peter wins first; Peter wins second.

2. Peter wins first; Paul wins second.

3. Paul wins first; Peter wins second.

4. Paul wins first; Paul wins second.

Paul wins the 64 pistoles in only one of the four cases; his share is therefore 16 pistoles.

Pascal replied that the answer was correct but the argument was faulty, or at least had proven unconvincing when he had made it to his friends, because it was not faithful to the rules of the game. By those rules, the players would play a second round only if Paul won the first. So there were only three ways play could go, not four.

Pascal preferred the argument represented by Figure 2 below. If Paul won the first game, Pascal argued, Paul would have an equal chance of winning 0 or 64, and this is worth 32. So at

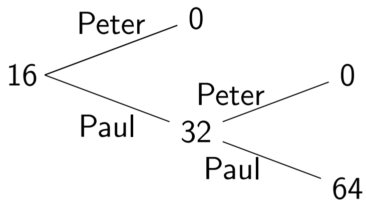

Figure 2: Game tree representing Pascal's solution

the outset, he has an equal chance of winning 0 or a position worth 32, and this is worth 16.

Neither argument was entirely novel. Fermat's logic of counting cases had been widely taught in Europe since the 13th century, and Pascal's argument has been found in manuscripts dating from the early 15th century. But the two arguments lead to different definitions of probability. Fermat's argument leads to the measure-theoretic definition developed by Maurice Fréchet, Andrei Kolmogorov, Doob, and Itô [10, 9, 15]: the probability of an event A is the total measure of the elementary events favoring A.[2] Pascal's argument leads to a game-theoretic definition: the probability of an event A is the initial stake needed to obtain 1 if A happens, 0 otherwise. Dividing the numbers in Figure 2 by 64, we see that we need 0.25 to get 1 if Paul wins both games.

The measure-theoretic and game-theoretic definitions are not in conflict. We can see this by considering Markov's inequality: if X is a nonnegative random variable with positive

[2]For a recent discussion of the counting of chances for three dice in the 13th century poem *De Vetula*, see [4]. For the prehistory of Kolmogorov's measure-theoretic axioms for probability, see [27].

expected value $\mathbf{E}(X)$, and $c > 0$, then

$$\mathbf{P}\{X \geq c\,\mathbf{E}(X)\} \leq \frac{1}{c}.$$

Because the payoff X cannot come out negative, you risk only $\mathbf{E}(X)$ when you pay this amount to buy X. So the inequality says that the probability of multiplying what you risk by c is no more than $1/c$.

An event has probability zero if you can arrive at 1 when it happens risking an arbitrarily small amount, or, equivalently, if you can arrive at ∞ risking a finite amount. A very small probability means you can multiply the capital risked by a very large number if the event happens. Here we are talking about all the money you risk, not merely your own money. The capital risked includes any line of credit on which you rely and, more generally, any money belonging to others that you risk.

When markets are incomplete, we may have only game-theoretic upper probabilities rather than game-theoretic probabilities. If no strategy delivers *exactly* 1 when A happens and 0 otherwise, then the upper probability of A is the initial stake needed to obtain *at least* 1 when A happens and 0 otherwise. This makes the game-theoretic definition very general.

3 Lévy on the intrinsic subjectivity of probability

Probability can be interpreted either objectively or subjectively. The measure-theoretic definition accommodates both interpretations and allows either to be taken to an extreme. We can accept the measure-theoretic definition while insisting, along with Karl Popper, that probability measures the propensity for events to happen, with no reference to any agent or observer.

We can equally well accept it while insisting, along with Bruno de Finetti, that probability measures only an agent's willingness to bet, with no objective content. The game-theoretic definition, on the other hand, has an intrinsic subjective element. A game requires players; betting requires bettors.

Paul Lévy explicitly recognized the subjectivity of probability. Along with Émile Borel, he contended that probability is initially subjective and can acquire an objective status only through experience. In his 1925 book on probability ([17], p. 3), he wrote,

> ... we have taken an essentially subjective point of view. The different cases are equally probable, because we cannot make any distinction among them.[3]

And in 1970 ([20], p. 206), a year before his death, he wrote,

> ... games of chance are to probability what solid bodies are to geometry, but with a difference. Solid bodies are given by nature, whereas games of chance were created to verify a theory imagined by the human mind, in such a way that the role of pure reason plays an even greater role in probability than in geometry.[4]

In his 1925 book, Lévy developed Jacques Hadamard's idea that probability theory is based on two fundamental notions:

[3] *In the original French:* ... nous nous sommes placés au point de vue essentiellement subjectif. Les différents cas possibles sont également probables parce que nous ne pouvons faire entre eux aucune distinction.

[4] *In the original French:* ... ce que les corps solides sont pour la géomtrie, les jeux de hasard sont pour le calcul des probabilités, mais avec une différence: les corps solides sont donnés par la nature, tandis que les jeux de hasard ont été créés pour vérifier une théorie imaginée par l'esprit humain, de sorte que le rôle de la raison pure est plus grand encore en calcul des probabilités qu'en géométrie.

1. equally probable events (*événements également probables*), and

2. event of very small probability (*événement très peu probable*).[5]

Whereas the notion of equally probable events expresses probability's subjective starting point, the notion of an event of very small probability allows us to connect probability to objective reality: we predict that the event will not happen. As Lévy further explained in his 1937 book ([18], p. 3),

> We can only discuss the objective value of the notion of probability when we know the theory's verifiable consequences. They all flow from this principle: a sufficiently small probability can be neglected. In other words: *a sufficiently unlikely event can in practice be considered impossible*.[6]

Borel called the principle that an event of sufficiently small probability can be considered impossible *the single law of chance*. Fréchet called it *Cournot's principle*.[7] Other authors who enunciated the principle in the mid-20th century include Kolmogorov

[5] Lévy devotes Chapter 1 to the first principle and Chapter 2 to the second. In the preface (p. viii), he cites a 1922 article [11] in which Hadamard stated the two principles.

[6] *In the original French:* Nous ne pouvons discuter la valeur objective de la notion de probabilité que quand nous saurons quelles sont les conséquences vérifiables de la théorie. Elles découlent toutes de ce principe: une probabilité suffisamment petite peut être négligée; en autre termes: *un événement suffisamment peu probable peut être pratiquement considéré comme impossible*.

[7] For a detailed discussion of the (non-Bayesian) subjectivism of the French school of probability, see notes 97, 105, and 145 of volume 1 of [5]. Cournot's own philosophy of probability was much more complex; see [6] and [21].

and Abraham Wald. In spite of the authority of these authors, the principle is often ridiculed on the grounds that whatever actually happens, when described with sufficient precision or detail, has exceedingly small probability. This ridicule overlooks the subjective aspect of probability that Lévy always emphasized. Even if they are given by a theory, probabilities can gain objective value only after they are adopted as subjective probabilities and used to make predictions. Prediction is meaningful only if it is limited and not self-contradictory. So from the infinitely many events to which a theory assigns small probability, we make a selection, predicting that certain salient events will not happen.[8] The theory gains objective status when these predictions succeed.

We can make this point more clearly by elaborating the betting picture that infused Lévy's work but often remained under the surface. In addition to a player who adopts the probabilities as beliefs, announcing a willingness to bet at the corresponding odds, consider a second player, who decides which of these betting offers to accept. We can assign this second player the task of deciding what predictions to make, because he decides what predictions to test. He tests a prediction by following a betting strategy that multiplies the capital he risks by a large factor when the prediction fails.

Consonant with this intrinsically subjective and game- theoretic view of probability, Lévy thought about a stochastic process in terms of its sample paths. Subjective probability is relative to knowledge, and the sample path records the evolution of the subject's knowledge. When we make the betting game

[8]This selection becomes relatively objective when there are only a few simple or salient events of small probability. If a probabilistic theory is expressed in a logical language, then there are only countably many probability-zero events we can describe, and we can predict that their union, which also has probability zero, will not happen. See [2].

explicit, the sample path becomes a partial description of a path through the game tree, picking out moves by a player who decides the outcomes.

On p. 360 of the second edition of his 1937 book, which appeared in 1954, Lévy discussed the relation between his own intuitive approach and Doob's unrelenting rigor. He wrote:

> A stochastic process is, in principle, a phenomenon in whose evolution chance intervenes at every instant. For Doob, a stochastic process is simply a random function $X(t)$ of a variable t, which one can assume represents time. [...] This notation is often convenient, though it presents as fully born in an instant what for me is essentially a perpetual becoming.[9]

Lévy's subjective and game-theoretic intuition also shines through his use of the notion of a martingale. A martingale is the capital process determined by a strategy for the player who tests the probabilities by making bets over time. Doob's unsparing reduction of probability to analysis, which followed a path already trod by Nobert Wiener in his mathematical treatment of Brownian motion and by Kolmogorov in his treatment of Markov processes [39, 14], obscured this intuition. Lévy, in contrast to all these authors, gloried in the martingale picture.

Lévy did not introduce this use of the word "martingale". It was introduced by Jean Ville in 1939. According to the dictionaries of the time (in French, English, and other European

[9] *In the original French:* Un processus stochastique est en principe un phénomène dans l'évolution duquel le hasard intervient à chaque instant. Pour Doob, un processus stochastique est simplement une fonction aléatoire $X(t)$ d'une variable t dont on peut imaginer quelle représente le temps. [...] C'est une notation souvent commode, bien qu'elle donne comme un tout né en un instant ce qui, pour moi, est essentiellement un perpétuel devenir.

languages), the martingale is an ill-advised strategy for betting: double your bet every time you lose. Ville used the word for any strategy for betting and then further used it to name the resulting capital processes. Lévy was not yet using the word in 1937, but he was using the idea. He emphasized his "condition $\mathcal{C}$",

$$\mathbf{E}_{t-1}(X_t) = 0,$$

which he explained in terms of betting: the rules of play for a given round may depend on the results of previous rounds, but they should be fair. The gain on the tth round, X_t, should have expected value zero just before the tth round. After Doob read and reviewed Ville's book, in 1940, both he and Lévy adopted Ville's terminology.

4 The game-theoretic intuition underlying Itô's stochastic calculus

In 1987, in the foreword to a volume of his papers ([30], p. xiii), Itô explained the relation between his own early work and Lévy's and Doob's work in these words:

> Having read A. Kolmogorov's *Grundbegriffe der Wahrscheinlichkeitsrechnung* (1933) I became convinced that probability theory could be developed in terms of measure theory as rigorously as in other fields of mathematics. In P. Lévy's book *Théorie de l'addition des variables aléatoires* (1937) I saw a beautiful structure of the sample paths of stochastic processes deserving the name of mathematical theory. From this book, I learned stochastic processes, Wiener's Brownian motion (Wiener process), Poisson process, and processes with independent incre-

> ments (differential processes). I was particularly interested in the decomposition theorem for differential processes, the core of this book. But I had a hard time following Lévy's argument because of his unique intrinsic description. Fortunately I noticed that all ambiguous points could be clarified by means of J. L. Doob's idea of regular versions presented in his paper "Stochastic processes depending on a continuous parameter" [*Trans. Amer. Math. Soc.* 42, 1938].[10] Checking Lévy's argument carefully from Doob's viewpoint, I was able to introduce Poisson random measures of jumps to really understand Lévy's spirit of the decomposition theorem.

What does Itô mean when he calls Lévy's description of stochastic processes intrinsic? What do Lévy and Itô see going on inside a stochastic process? They see the sample path. At every instant, a gambler bets and chance intervenes to move the path and the gambler's martingale along.

As Shinzo Watanbe has explained ([38], pp. 1–2), Itô's astonishingly original translation of Lévy's sample-path intuition into Doob's measure-theoretic framework was already well launched in 1942:

> Although the study of modern probability theory in Japan certainly started before 1940, the war disrupted communications with other advanced countries. Under these circumstances, Itô completed two important contributions [12, 13] that are now considered the origin of *Itô's stochastic analysis* or *Itô's stochastic calculus*. In the first work, he gave

[10] The issue of the journal in which this article appeared was actually dated July 1937 [8].

> a rigorous proof of what is now called the Lévy-Itô theorem for the structure of sample functions of Lévy processes, through which we have a complete understanding of the Lévy-Khinchin formula for canonical forms of infinitely divisible distributions. In the second work, he developed a complete theory of stochastic differential equations determining sample functions of diffusion processes whose laws are described by Kolmogorov's differential equations. In this work, he introduced the important notion of a stochastic integral and the basic formula now known as *Itô's formula* or *Itô's lemma* and thus founded a kind of Newton-Leibniz differential and integral calculus for a class of random functions now often called *Itô processes*.

The only other mathematician who achieved an early mastery of Lévy's pathwise picture comparable to Itô's was Lévy's student Wolfgang Doeblin, who perished as Nazi Germany overran northern France in 1939. Doeblin's anticipation of Itô's stochastic calculus came to light only in the year 2000 [3].[11]

Itô's viewpoint has been vindicated not only by its success as pure mathematics but also by the fit between its intrinsic, essentially game-theoretic vision and the project of providing a mathematical foundation for finance. In finance, Itô's integral

$$\int H(X)\,dX$$

is understood as the capital process resulting from holding at each point in time $H(X)$ units of a security or portfolio that has the price process X. But as often happens with applications of probability, the success of the vision has outrun the intuition.

[11] Doeblin did not succeed in defining an Itô integral, but see [22, 23].

The integrator X is a process, not just a path of a process, and the integrand H is also a process — a complete strategy for trading in X over time, not merely the actual trading along a particular path for X. This is more than a verbal quibble, as Itô's definition of the integral requires only convergence in probability, whereas an understanding in terms of paths would seem to require convergence for almost all paths — that is, convergence with probability one.

This tension between game-theoretic intuition and measure-theoretic mathematics is not easily resolved. The price of a financial security follows only one path, and a trader can create his own capital path by varying how much of the security he holds over time without adopting a complete strategy for how he will invest if the security's price evolves differently. Any mathematical theory of trading in continuous time will be a very idealized picture of this reality, and we would expect it to involve various topological and analytical conditions on the paths — on the path taken by the price of the security and on the path taken by the trader's holdings. But why must probability be involved? Why can we not define Itô's integral in a probability-free manner?

The best known effort to address this question is that of Hans Föllmer, who showed in 1981 that Itô's integral can be defined without probabilistic assumptions provided that H is a smooth function and the price path X has finite quadratic variation with respect to a particular sequence of finer and finer partitions of the time interval, on which the value of the integral then depends.

I cannot review all the work towards a probability-free Itô integral that has followed Föllmer's. But I want to put forward the claim that game-theoretic probability advances our understanding of the possibilities. Before discussing this point in continuous time, let us look more carefully at game-theoretic probability in discrete time.

5 Game-theoretic probability in discrete time

Pascal did not have our modern mathematical theory of games. But we do, and so we can make his game-theoretic foundation mathematically precise. This involves defining the game precisely. What are the rules of play? Who are the players? What information do the players have? What are the rules for winning?

Consider a game with three players, Forecaster, Skeptic, and Reality, who play in order as follows:

$\mathcal{K}_0 = 1$.
For $n = 1, 2, \ldots$:
  Forecaster announces $p_n \in [0, 1]$.
  Skeptic announces $s_n \in \mathbb{R}$.
  Reality announces $y_n \in \{0, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + s_n(y_n - p_n)$.

Each player sees the others' moves as they are made, and a player may also receive other information before or during the game. The game pits Skeptic against the other two players. Skeptic wins if these two conditions are both satisfied:

1. $\mathcal{K}_n \geq 0$ for all n.

2. As $n \to \infty$, either $\mathcal{K}_n \to \infty$ or $\frac{1}{n}\sum_{i=1}^{n}(y_i - p_i) \to 0$.

Reality decides the outcomes. Why "Reality" and not "Nature"? Because we do not want to assert that the outcomes are determined by laws of nature. Our mathematical results do not require Reality to follow any law. Reality can do whatever she wants.

On each round, Reality announces either a 0 or a 1, which can encode heads or tails, yes or no. Forecaster's move p_n can be understood as a probability for Reality announcing y_n to be 1. Skeptic is the one who bets, and $\mathcal{K}_0, \mathcal{K}_1, \ldots$ is his capital process. This capital changes on each round. If Reality announces 1, Skeptic adds $s_n(1-p_n)$ to his capital. If Reality announces 0, Skeptic adds $-s_n p_n$. In order to win, Skeptic cannot risk more than his initial unit capital. If his capital $\mathcal{K}_n$ becomes negative, he loses. Provided that he keeps $\mathcal{K}_n$ nonnegative, he can win in either of two ways: either (1) he becomes infinitely rich, or (2) the difference between the average outcome and the average probability tends to 0. This is the game-theoretic strong law of large numbers.

In Borel's measure-theoretic strong law of large numbers, convergence happens except on a set of probability 0. Here it happens unless Skeptic gets infinitely rich. Turning 1 into infinity becomes the definition of the probability 0. But the game-theoretic result is not merely a translation from measure theory; it is a theorem in game theory. When Forecaster and Reality play against Skeptic as a team, the game is a perfect information game with two players. So we know from Martin's generalization of Zermelo's theorem[12] that one of the players has a winning strategy, and we can prove that it is Skeptic.

As shown in *Probability and Finance*, many limit theorems in probability theory can be interpreted and restated in this game-theoretic manner. Whereas the measure-theoretic versions of

[12] Zermelo's theorem says that in a two-player perfect-information game that ends after a finite number of rounds and always has a winner, one of the players has a winning strategy. Martin's theorem generalizes this result to games in which the winner may depend on an infinite sequence of play. For references, see *Probability and Finance* [26], pp. 94–98.

these theorems say that certain events happen except on a set of outcomes that has small or zero probability, the game-theoretic version says that Skeptic has a strategy that multiplies the capital he risks by a large or infinite factor if these events do not happen. The game-theoretic proofs are constructive: they construct the strategy for Skeptic.

To illustrate this constructivity, consider this simplification of our three-player example. Here p_n is always $\frac{1}{2}$, and Forecaster is removed from the protocol because he no longer has a role to play.

$\mathcal{K}_0 = 1.$
For $n = 1, 2, \ldots$:
  Skeptic announces $s_n \in \mathbb{R}$.
  Reality announces $y_n \in \{0, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + s_n(y_n - \frac{1}{2})$.

Here, as Jean Ville noted ([7], Section 3.3), one winning strategy for Skeptic is for him to choose his s_n by the formula

$$s_n := \frac{4\,\mathcal{K}_{n-1}}{n+1}\left(r_{n-1} - \frac{n-1}{2}\right),$$

where

$$r_{n-1} := \sum_{i=1}^{n-1} y_i.$$

This strategy tracks how far Reality's relative frequency of 1s is from $\frac{1}{2}$ and always bets on Reality moving it farther in the same direction. We can show by induction that Skeptic's capital from the strategy is

$$\mathcal{K}_n = 2^n \frac{r_n!(n - r_n)!}{(n+1)!}.$$

From the assumption that this remains bounded, you can show that the proportion of 1s, r_n/n, converges to $\frac{1}{2}$. (Approximate the factorials using Stirling's formula and apply the Kullback–Leibler inequality.)

We can also develop abstract game-theoretic versions of probability's classical limit theorems. Here the players' move spaces are not fully specified, just as probability spaces and random variables are often not fully specified in measure-theoretic probability. Abstract game-theoretic probability uses a concept of upper expectation, similar to the concept of *upper prevision* used in the theory of imprecise probability [37, 1]. Whereas a probability measure's expectation operator is linear, an upper expectation is only subadditive and positively homogeneous.

Here is the axiomatic definition of upper expectation used in *Game-Theoretic Foundations for Probability and Finance*: Suppose $\mathcal{Y}$ is a nonempty set, and let $\mathcal{G}$ be the set of all mappings from $\mathcal{Y}$ to $[-\infty, \infty]$. A mapping $\overline{\mathbf{E}}$ from $\mathcal{G}$ to $[-\infty, \infty]$ is an *upper expectation on* $\mathcal{Y}$ if it satisfies these five axioms:

1. If $f_1, f_2 \in \mathcal{G}$, then $\overline{\mathbf{E}}(f_1 + f_2) \leq \overline{\mathbf{E}}(f_1) + \overline{\mathbf{E}}(f_2)$.
2. If $f \in \mathcal{G}$ and $c \in (0, \infty)$, then $\overline{\mathbf{E}}(cf) = c\overline{\mathbf{E}}(f)$.
3. If $f_1, f_2 \in \mathbf{G}$ and $f_1 \leq f_2$, then $\overline{\mathbf{E}}(f_1) \leq \overline{\mathbf{E}}(f_2)$.
4. For each $c \in [-\infty, \infty]$, $\overline{\mathbf{E}}(c) = c$.
5. If $0 \leq f_1 \leq f_2 \leq \cdots \in \mathcal{G}$, then $\overline{\mathbf{E}}\left(\lim_{k\to\infty} f_k\right) = \lim_{k\to\infty} \overline{\mathbf{E}}(f_k)$.

We interpret $\overline{\mathbf{E}}(f)$ as the price for the payoff f. The first four axioms are the most essential. The fifth is a weakening of measure-theoretic probability's axiom of countable additivity or continuity. It is often unneeded, but it sometimes simplifies the theory.

With the notion of an upper expectation, we can formulate abstract perfect-information protocols involving our three players. Here is an example:

$\mathcal{K}_0 = 1$.
FOR $n = 1, 2, \ldots$:
  Forecaster announces an upper expectation $\overline{\mathbf{E}}_n$ on $\mathcal{Y}$.
  Skeptic announces $f_n \in \mathcal{G}$ such that $\overline{\mathbf{E}}_n(f_n) \leq \mathcal{K}_{n-1}$.
  Reality announces $y_n \in \mathcal{Y}$.
  $\mathcal{K}_n := f_n(y_n)$.

Game-Theoretic Foundations for Probability and Finance shows in detail how this framework can be used to develop classical discrete-time probability. Using an abstract protocol of the type just described, we can derive a global upper expectation that gives prices for variables that depend on the entire sequence of moves by Skeptic's opponents. This global upper expectation can be used to state and derive generalizations of classical limit theorems, including Lévy's zero-one law, which says that the global expected value of a variable tends almost surely to the variable's actual value. We can then develop new insights and applications, including Jeffreys' law (two successful Forecasters will produce consistent forecasts in the long run) and defensive forecasting (Forecaster has strategies that will resist Skeptic's statistical tests and produce decisions with good long-run properties, regardless of how Reality plays).

Many of the results reported in *Game-Theoretic Foundations for Probability and Finance* have been developed or inspired by the Japanese school of game-theoretic probability, led since 2003 by Kei Takeuchi (Professor Emeritus of the University of Tokyo and currently at Meiji Gakuin University) and Akimichi Takemura (also Professor Emeritus of the University of Tokyo and now the director of data science education and research at Shiga University). In 2004, Takeuchi published a book on game-theoretic probability [31]. Takemura played a key role in the development of defensive forecasting and the game-theoretic zero-one law [29]. Work by Takeuchi and Takemura, in collaboration with Masayuki Kumon [32], helped launch the approach to continuous-time game-theoretic probability presented in *Game-Theoretic Foundations for Probability and Finance*. To this I now turn.

6 Game-theoretic probability in continuous time

How do you bet in continuous time? *Probability and Finance* used nonstandard analysis to formalize the idea that a trader can hold a different portfolio in every successive infinitesimal period of time. This approach can be developed further, in many different ways no doubt, but a more conventional idealization, closer to Itô's and perhaps more revealing, can be developed using standard analysis. In this idealization, we combine trading strategies that change their portfolio with greater and greater frequency, and then enlarge the resulting class of capital processes further by closing it under various limiting processes. The class of elementary trading strategies with which we begin was studied by Vovk in 1993 [33]. In a study that first appeared as a working paper in 2007 [32], Takeuchi and his colleagues enlarged this class of strategies by considering countably many stopping times. In subsequent papers, Vovk showed that variations on this idea lead to results analogous to Dubins and Schwarz's reduction of continuous martingales to Brownian motion via a time change. Nicolas Perkowski and David Prömel then argued for tightening the resulting notion of upper probability using liminf, so that there are more null sets, in work that first appeared as a working paper in 2015 [24]. Vovk subsequently proposed enlarging the class of capital processes directly using liminf, in a bold way that may prove definitive and is presented in detail in *Game-Theoretic Foundations for*

Probability and Finance.[13]

Suppose we divide a unit of capital, assumed to be infinitely divisible, among countably many accounts and use each account to trade at discrete points in time. If the successive accounts use finer and finer grids of trading points, then this looks like trading in continuous time. If we fix a strategy for the trading, we have a game in normal form: the trader moves first, announcing a trading strategy, and then the market moves, giving a price path. As it turns out, the assumption that the trader cannot multiply his capital infinitely leads to the conclusion that X's path looks like Brownian motion modulo a time change. In fact, the trader has strategies that make him infinitely rich instantaneously, not merely in the long run, as soon as Brownian properties, such as the absence of isolated zeros and the absence of monotonicity, are violated.[14] This is a game-theoretic version of the Dubins–Schwarz theorem. Unlike the usual Dubins–Schwarz theorem, it is probability-free in the sense that no probabilities are assumed. The Brownian properties hold with game-theoretic probability one, but this probability is defined by the game; it means only that the trader can multiply the capital he risks infinitely if the properties do not hold.

One Brownian property guaranteed by the Dubins–Schwarz theorem is the existence of quadratic variation. This seems to open the way to the development of a "probability-free" game-theoretic version of Itô's stochastic calculus via Föllmer's result. To make this work, however, and to eliminate the dependence of Föllmer's construction on a particular sequence of partitions, it seems necessary to further idealize — in a rather nonconstruc-

[13] Many references are given in *Game-Theoretic Foundations for Probability and Finance*. See especially [33, 24, 36].

[14] See [34] and Chapter 13 of *Game-Theoretic Foundations for Probability and Finance*. Analogous but more complicated results can be developed in discrete time; see [35].

tive way — what the trader can accomplish in continuous time. The key idea, due to Vovk, is to close under liminf the set of nonnegative supermartingales obtained from trading strategies. (A supermartingale is a capital process from a trading strategy that may waste money; it is a martingale only if no money is wasted.) This enlarged set of nonnegative supermartingales can be used to define the notion of instant enforcement. (A property holds *with instant enforcement* if some nonnegative supermartingale becomes infinite as soon as the property fails.) The class of continuous martingales can then be enlarged by taking limits that are uniform over compacts modulo instant enforcement. This leads to a purely game-theoretic version of the stochastic calculus for continuous martingales (and semi-martingales, a semimartingale being the sum of a martingale and a finite-variation process), in which Itô's formula and other results hold with instant enforcement.

7 Conclusion

Vovk's game-theoretic foundation for continuous-time stochastic processes formalizes Lévy's intrinsic pathwise intuition. Because game-theoretic probability is consistent with and complementary to measure-theoretic probability, it can also be seen as an elaboration of Itô's stochastic calculus.

Because this game-theoretic picture involves only emergent probabilities — probabilities that are determined by trading opportunities — it can help us disentangle continuous-time financial mathematics from the philosophical controversies and confusions surrounding probability theory. It is my hope that it will clarify the applicability of the stochastic calculus to finance and the limits of that applicability. Closing a class of continuous-time capital processes under liminf is a highly nonconstructive step; it is equivalent to a process of transfinite induction. This puts us on notice that we are dealing with an extreme idealization.[15] When it brings clarity and simplicity, such an idealization can provide great insights, but the accuracy of these insights in practical problems needs to be measured and tested with great care. We can use the game-theoretic picture not only to derive these insights but also to measure their accuracy as simplified approximations to discrete-time realities. Rather than pretend that the prices we see in financial markets are glimpses of an underlying continuous-time reality, we can ask how closely actual financial games in discrete time can bring us to the continuous-time idealization.

In the wake of the 2007 financial crisis, mathematicians debated their responsibility. Have we since found a way to combine

[15] To underline the point that a stochastic process in continuous time is alway an extreme idealization, we can do no better than to quote Lévy once again ([19], p. 286):

> ...in a concrete case, in the case of Brownian motion for example, nature actualizes only at the microscopic level phenomena that the mathematician, to facilitate his research, takes all the way down to the infinitely small. We think, moreover, even though some eminent scientists have a different opinion since the work of Heisenberg, that the notion of chance is a notion that the scientist introduces because it is convenient and productive but that nature ignores.
>
> *In the original French:* ...dans un cas concret, dans celui du mouvement brownien, par exemple, la nature réalise seulement à l'échelle microscopique des phénomènes que le mathématicien, pour la commodité de ses recherches, prolonge jusqu'à l'infiniment petit. Nous pensons, d'ailleurs, quoique depuis les travaux d'Heisenberg d'éminents savants ne soient pas de cet avis, que la notion de hasard est une notion que le savant introduit parce qu'elle est commode et féconde mais que la nature ignore.

teaching the stochastic calculus with teaching its limitations? Surely the game-theoretic framework can advance that project.

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