« Quadratic » Hawkes processes:
A microfoundation for rough vol models?

Fat-tails and Time Reversal Asymmetry

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(building on previous work with

Rémy Chicheportiche & Steve Hardiman)

« Stylized facts »

I. Well known:

- Fat-tails in return distribution $p(r) \underset{|r|\to\infty}{\sim} \frac{C'}{|r|^{1+\nu}}$

with a (universal?) exponent ν around 4 for many different assets, periods, geographical zones,…

- Fluctuating volatility with « long-memory » -- log-vol is close to a fBM with H small (« rough vol »*), possibly H=0 (« multifractal »**)

- Leverage effect (negative return/vol correlations)

* Gatheral-Jaisson-Rosenbaum; ** Bacry-Muzy

« Stylized facts »

II. The Zumbach effect:

- Intuition: past trends, up or down, increase future vol more than alternating returns (for a fixed HF activity/volatility)
- Reverse not true (HF vol does not predict more trends)

→ Can one create « micromodels » that capture all these effects?

→ What is the large scale limit of these models? « Rough » volatility?

Hawkes processes

- A *self-reflexive feedback* framework, mid-way between purely stochastic and agent-based models
- Activity is a Poisson Process with history dependent rate:

$$\lambda_t = \lambda_\infty + \int_{-\infty}^{t} \phi(t-s)\ \mathrm{d}N_s$$

- Feedback intensity $n \equiv \int_0^\infty \phi(\tau) d\tau$ < 1
- Calibration on financial data suggests *near criticality* (n → 1) and *long-memory* power-law kernel ϕ : the « Hawkes without ancestors » limit (Brémaud-Massoulié)

Continuous time limit of near-critical Hawkes

- Jaisson-Rosenbaum show that when n → 1 Hawkes processes converge (in the right scaling regime) to either:

  i) Heston for short-range kernels
  ii) Fractional Heston for long-range kernels, with a small Hurst exponent H

- But: still no fat-tails and no TRA…
- J-R suggest results apply to log-vol, but why?
- Calibrated Hawkes processes generate very little TRA, even on short time scales (see below)

Generalized Hawkes processes

- Intuition: not just past activity, but *price moves themselves* feedback onto current level of activity
- The most general quadratic feedback encoding is:

$$\lambda_t = \lambda_\infty + \frac{1}{\psi}\int_{-\infty}^{t} L(t-s)\,\mathrm{d}P_s + \frac{1}{\psi^2}\int_{-\infty}^{t}\int_{-\infty}^{t} K(t-s,t-u)\,\mathrm{d}P_s\,\mathrm{d}P_u$$

- With: $dN_t := \lambda_t\, dt$; $dP := (+/-)\,\psi\, dN$ with random signs

- L(.): leverage effect neglected here (small for intraday time scales)
- K(.,.) is a symmetric, positive definite operator
- Note: $K(t,t)=\phi(t)$ is exactly the Hawkes feedback ($dP^2=dN$)

Generalized Hawkes processes

$$\lambda_t = \lambda_\infty + \frac{1}{\psi}\int_{-\infty}^{t} L(t-s)\,\mathrm{d}P_s + \frac{1}{\psi^2}\int_{-\infty}^{t}\int_{-\infty}^{t} K(t-s,t-u)\,\mathrm{d}P_s\,\mathrm{d}P_u$$

- 2- and 3-points correlation functions

-
$$\mathcal{C}(\tau) \equiv \mathbb{E}\left[\frac{\mathrm{d}N_t}{\mathrm{d}t}\frac{\mathrm{d}N_{t-\tau}}{\mathrm{d}t}\right] - \overline{\lambda}^2 = \mathbb{E}\left[\lambda_t \frac{\mathrm{d}N_{t-\tau}}{\mathrm{d}t}\right] - \overline{\lambda}^2,$$

$$\mathcal{D}(\tau_1,\tau_2) \equiv \frac{1}{\psi^2}\,\mathbb{E}\left[\frac{\mathrm{d}N_t}{\mathrm{d}t}\frac{\mathrm{d}P_{t-\tau_1}}{\mathrm{d}t}\frac{\mathrm{d}P_{t-\tau_2}}{\mathrm{d}t}\right] = \frac{1}{\psi^2}\,\mathbb{E}\left[\lambda_t \frac{\mathrm{d}P_{t-\tau_1}}{\mathrm{d}t}\frac{\mathrm{d}P_{t-\tau_2}}{\mathrm{d}t}\right]$$

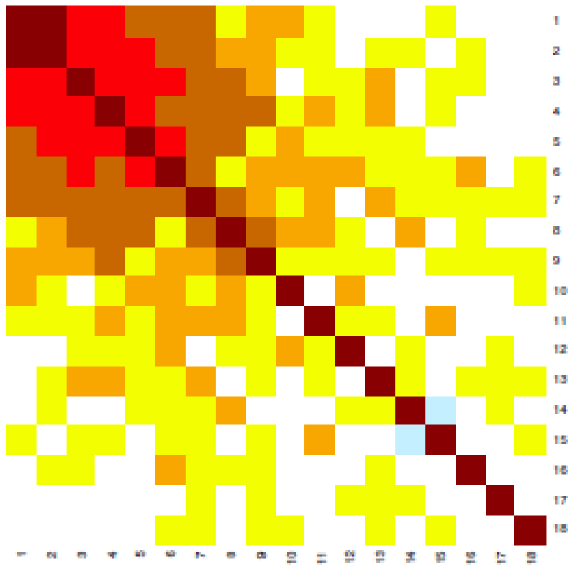
$$\mathcal{C}(\tau) = \kappa\overline{\lambda}K(\tau,\tau) + \int_{-\infty}^{\tau}\mathrm{d}u\,K(\tau-u,\tau-u)\mathcal{C}(u) + 2\int_{0^+}^{\infty}\mathrm{d}u\int_{u^+}^{\infty}\mathrm{d}r\,K(\tau+u,\tau+r)\mathcal{D}(u,r),$$

- And a similar *closed* equation for $\mathcal{D}(.,.)$, $\mathcal{C}(.)$

- This allows one to do a GMM calibration

Calibration on 5 minutes US stock returns

- Using GMM as a starting point for MLE, we get for K(s,t):

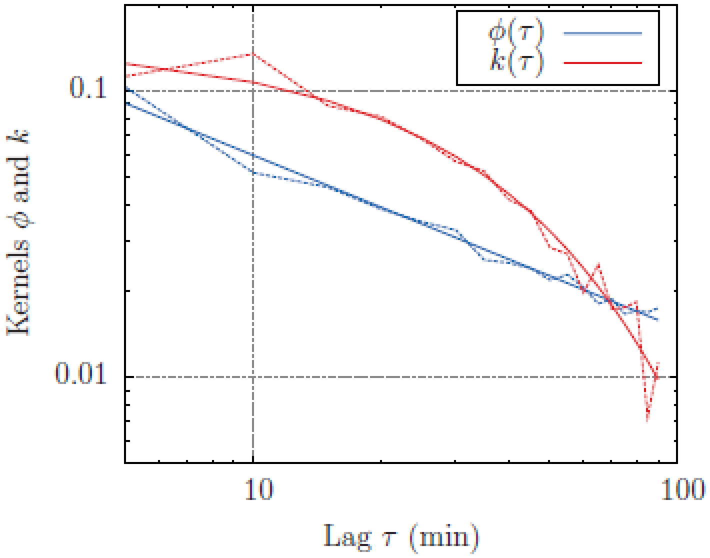


- K is well approximated by Diag + Rank 1:

$$K(\tau, \tau') \approx \phi(\tau)\delta_{\tau-\tau'} + k(\tau)k(\tau')$$

Calibration on 5 minutes US stock returns

$$K(\tau,\tau') \;\approx\; \phi(\tau)\delta_{\tau-\tau'} + k(\tau)k(\tau')$$



$$\phi(\tau) = g\tau^{-\alpha} \quad , \quad k(\tau) = k_0 \exp(-\omega\tau),$$

$$g = 0.09,\;\; \alpha = 0.60,\;\; k_0 = 0.14,\;\; \omega = 0.15.$$

Generalized Hawkes processes: Hawkes + « ZHawkes »

$$K(\tau,\tau') \;\approx\; \phi(\tau)\delta_{\tau-\tau'} + k(\tau)k(\tau')$$

$$\lambda_t = \lambda_\infty + H_t + Z_t^2,$$

$$H_t := \int_{-\infty}^{t} \phi(t-s)\,\mathrm{d}N_s, \qquad Z_t = \frac{1}{\psi}\int_{-\infty}^{t} k(t-s)\,\mathrm{d}P_s.$$

Z_t : moving average of price returns, i.e. recent « trends »

→ The Zumbach effect: trends increase future volatilities

The Markovian Hawkes + ZHawkes processes

$$\lambda_t = \lambda_\infty + H_t + Z_t^2,$$

$$H_t := \int_{-\infty}^{t} \phi(t-s)\, \mathrm{d}N_s, \qquad Z_t = \frac{1}{\psi}\int_{-\infty}^{t} k(t-s)\, \mathrm{d}P_s.$$

With: $k(t) = \sqrt{2 n_Z \omega}\, \exp(-\omega t)$ and $\phi(t) = n_H \beta \exp(-\beta t)$

In the continuum time limit: ($h = H$; $y = Z^2$):

$$dh = [- (1 - n_H)\, h + n_H\, (\lambda + y)\,]\, \beta\, dt$$

$$dy = [- (1 - n_Z)\, y + n_Z\, (\lambda + h)\,]\, \omega\, dt + [2\, \omega\, n_Z\, y\, (\lambda + y + h)]^{1/2}\, dW$$

→ 2-dimensional generalisation of Pearson diffusions ($n_H = 0$)

→ The y process is asymptotically multiplicative, as assumed in many « log-vol » models (including Rough vols.)

The Markovian Hawkes + ZHawkes processes

$$dh = [-(1 - n_H)\, h + n_H\, (\lambda + y)]\beta\, dt$$

$$dy = [-(1 - n_Z)\, y + n_Z\, (\lambda + h)]\omega\, dt + [2\, \omega\, n_Z\, y\, (\lambda + y + h)]^{1/2}\, dW$$

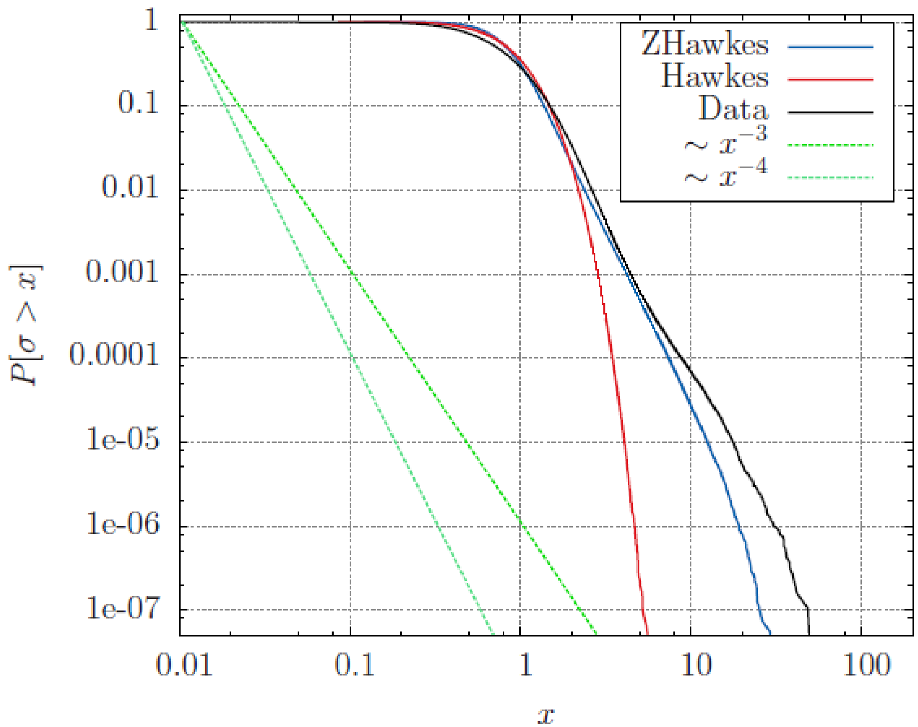
→ The upshot is that the vol/return distribution has a power-law tail with a computable exponent, for example:

$$\beta >> \omega \rightarrow \nu = 1 + (1 - n_H)/n_Z$$

→ Even when n_Z is smallish, n_H conspires to drive the tail exponent ν in the empirical range ! – see next slide

→ Note: $n_Z > 1$ defines a stationary Hawkes process with infinite mean intensity!
(C. Aubrun, M. Benzaquen, JPB)

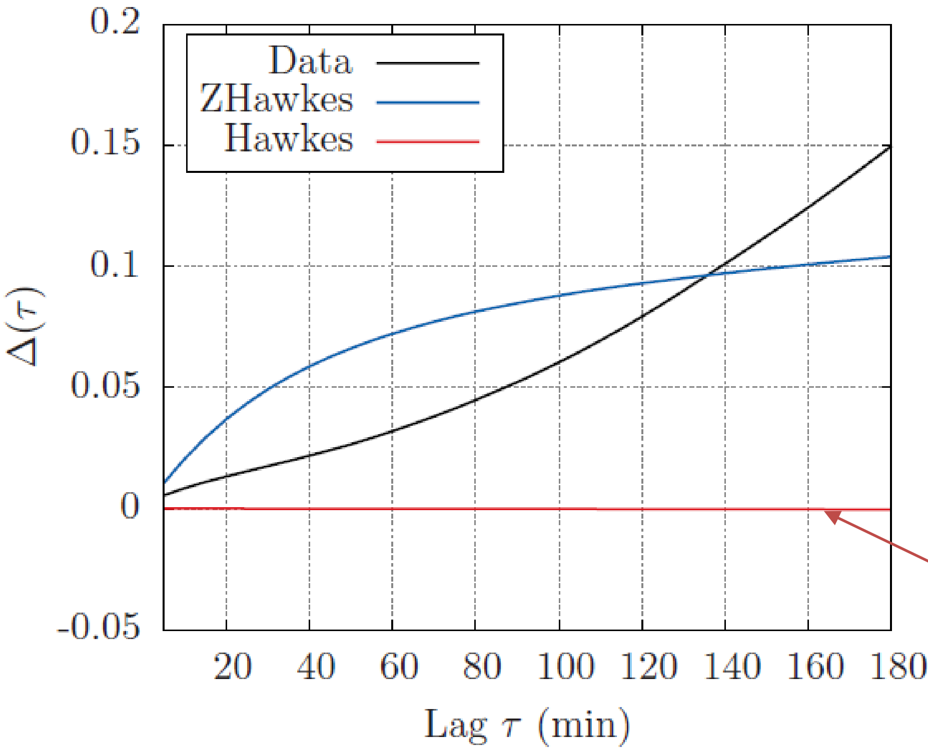
The calibrated Hawkes + ZHawkes process: numerical simulations



Fat-tails are indeed accounted for with $n_Z = 0.06$!

Note: $\Delta P_\tau = \pm\psi$ so tails ***do not*** come from « residuals »

The calibrated Hawkes + ZHawkes process: numerical simulations



$$\Delta(\tau) = \frac{\sum_{\tau'=1}^{\tau} [C(\tau') - C(-\tau')]}{2 \sum_{\tau'=1}^{q} \max(|C(\tau')|, |C(-\tau')|)}$$

where C is the cross-correlation between σ_{HF} and |r|

Close to zero!

The level of TRA is also satisfactorily reproduced

(wrong concavity probably due to intraday non-stationarities not accounted for here)

Generalisation: order book activity & liquidity crises

* Fosset, JPB, Benzaquen, 2021

6-vectors — 6 x 6 matrix — 6-vectors

$$\boldsymbol{\lambda}_t = \boldsymbol{\alpha}_0 + \int_0^t \boldsymbol{\phi}(t-s)\, d\boldsymbol{N}_s + \int_0^t \boldsymbol{L}(t-s)\, dP_s + \int_0^t \int_0^t \boldsymbol{K}(t-s, t-u)\, dP_s dP_u$$

- Consider the two best limits and 6 event-types: MO, LO, CA, described by a 6-dimensional rate vector λ_t (→ 3 by symmetry)
- These rates depend on past events d**N** and past price changes dP
- The second term is a Hawkes feedback (bid/ask symmetric)
- The third term is a « leverage » feedback (bid/ask antisymmetric)
- The last term couples past volatility K(u,u) and past trends K(u,v) to present rates (bid/ask symmetric) – cf. the Zumbach effect

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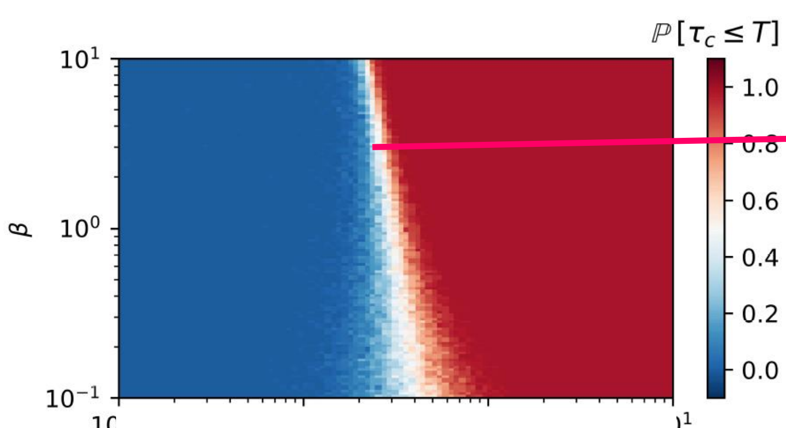
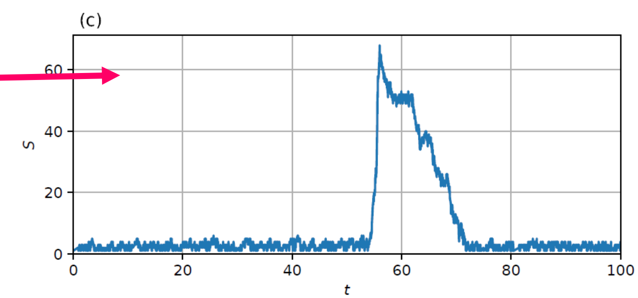
$$\boldsymbol{\lambda}_t = \boldsymbol{\alpha}_0 + \int_0^t \boldsymbol{\phi}(t-s)\,d\boldsymbol{N}_s + \int_0^t \boldsymbol{L}(t-s)\,dP_s + \int_0^t\int_0^t \boldsymbol{K}(t-s,t-u)\,dP_s dP_u$$

- Calibration on tick by tock data shows a clear influence of past trends <u>and</u> past volatility on event rates, which <u>decrease</u> the volume in the order book → a possible destabilising feedback loop!



Strength of feedback



Spread dynamics

Conclusion

- Generalized Hawkes Processes: a natural extension of Hawkes processes accounting for « trend » (Zumbach) effects on volatility – a step to close the gap between ABMs and stochastic models
- Leads naturally to a multiplicative process for volatility
- Accounts for tails (induced by micro-trends) and TRA
- Adding the « Zumbach » term in Rough Vol. models leads to a very satisfactory model → see M. Rosenbaum

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- A lot of work remaining (empirical and mathematical)
- Multivariate generalisation (C. Aubrun & M. Benzaquen)
- Real « Micro » foundation ? Higher order terms ?